The Role of the Second Prize in All-Pay Auctions with Two Heterogenous Prizes

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Abstract

We study complete information all-pay contests with n players and two heterogeneous prizes with distinct values. Among the players, $n-1$ are symmetric (i.e., they evaluate the prizes in a similar manner), whereas the remaining player has different valuations for each of the prizes. Our analysis focuses on the equilibrium profiles and expected payoffs for the case of three players. We also partially extend our analysis for cases with additional players. Our results show that in all-pay auctions with heterogeneous prizes, the ordering of the players according to their expected payoffs in equilibrium might vary significantly, depending on both prizes. Furthermore, although the values for the first (larger) prize have the greatest effect on the identity of the players with positive expected payoffs, the value of the second prize might have a non-negligible effect.

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1 Introduction

Contests in which multiple prizes are awarded are quite ubiquitous. Examples include employees who exert effort for the purpose of promotions in organizational hierarchies, students who compete over grades (and the adjacent ranking), political competitions for ranked places in parliamentary systems, and obviously sports events where athletes compete over medals or various monetary prizes. Such contests with multiple prizes can be modeled in several ways, one of the most well-known being the all-pay auction.[1] In this contest form, the players with the highest bids receive the prizes, but all the players, including those who win nothing, bear the costs of their bids.

At present, most of the contest literature has focused on single-prize all-pay auctions where the highest bidder is awarded the prize (known as the winner-take-all contest), whereas studies concerning all-pay auctions with multiple prizes, especially heterogenous ones, are rather neglected. The reason for this is quite straightforward - there is a substantial difference, in terms of complexity, between the analysis of a single-prize all-pay auction or even one with several identical prizes, and that of an all-pay auction with heterogenous prizes. For example, in a complete information single-prize contest, the player with the highest valuation wins the prize with the highest probability and has the highest expected payoff. Moreover, if one player has a strictly higher valuation for winning compared to all the other players, then only he has a positive expected payoff, while all the others have an expected payoff of zero (see Baye et al. 1996). Likewise, in a complete information all-pay auction with $k \geqslant 2$ identical prizes, the players with the k highest values gain positive expected payoffs such that a higher private value entails a higher expected payoff.

In contrast, when there are at least two heterogeneous prizes and the ordering of the players' valuations vary across prizes, the identity of the winners for each of the prizes, as well as the order of the players' expected payoffs, are ambiguous. A priori, it is unclear how one should evaluate the winning probability and expected payoff of a player with a high value for the first prize and a low value for the second one compared to those of a player with a lower value for the first prize and a higher value for the second one.

In this paper, we try to shed light on the players' behavior in all-pay auctions with heterogenous prizes (i.e., the players' valuations vary across prizes) by emphasizing the importance of all prizes, including the prizes with the smaller values. We assume that each player has a higher value for the first prize than for the second one, but the player with the highest value for each prize is not necessarily the same. To illustrate such a situation, consider a hierarchical workplace where several junior employees compete for a promotion to two higher-ranked positions for a manager, one at the headquarters in New York and the other at a branch office in Tel-Aviv. Suppose the former promises a higher salary, so that the position at the headquarters is more valuable to each employee. Then, while an American employee values the first position the most, it would be an Israeli candidate who values the second the most.

In order to deal with the players' behavior in our complex environment, we assume that each of the n players is one of two types: there are $n-1$ symmetric players, all of whom value each prize

[1] See, among others, Hillman and Samet (1987), Hillman and Riley (1989), Baye et al. (1993), Amman and Leininger (1996), Krishna and Morgan (1997), Che and Gale (1998), Lizzeri and Persico (2000), Siegel (2009), Sela (2012), Hart (2016), Einy et al. (2017), and Lu and Parreiras (2017).

similarly (though the values of the first and second prize do differ), while the remaining player has different values for both prizes compared to his opponents (note that all valuations are public). The contest evolves as follows. First, each player chooses a bid. Next, the player with the highest bid wins the first prize, and the player with the second highest bid wins the second prize. All players bear the cost of their bids, independently of their winning status. It turns out that the most complex scenario in our model is when there are only three players, namely, two symmetric players and a single asymmetric one. The rationale is that if there are more than two symmetric players, in any equilibrium, the direct competition reduces their expected payoff to zero since the number of prizes is smaller than the number of the symmetric players. On the other hand, if there are only two symmetric players, they might have positive expected payoffs. Thus, we mostly focus on three players, while providing some generalizations for $n > 3$ players.

In contrast to the equilibrium profiles in the all-pay auction with a single prize in which the players' efforts (or bids) are derived from a common support, in the all-pay auction with two heterogenous prizes, they are not necessarily derived from the same support. Moreover, the supports of the players' strategies are not necessarily convex, namely, they include gaps such that the players' mixed strategies (distributions over bids) are not strictly increasing.[2] Thus, we divide our analysis into five cases (A-E) according to the relations between the players' values for the prizes, and for each case we derive the players' equilibrium bids. Since we provide an explicit solution of the players' equilibrium strategies, we are able to calculate the players' expected payoffs as well.

Finding the equilibrium strategies in all-pay auctions with complete information is not an easy task since one first needs to guess their structure, and then confirm that they satisfy the required conditions. Thus, we first carried out numerous simulations. Then, we guessed the forms of the equilibrium strategies and proved that they satisfy the sufficient equilibrium conditions. We found that the profiles derived from the indifference conditions are not necessarily strictly increasing, and therefore it is difficult to prove their uniqueness. However, our results (Lemmas 1-7) characterize the necessary properties of the equilibrium strategies. These properties define "almost" uniquely the equilibrium strategies, and moreover, they define unambiguously the form of the equilibrium in each of the cases A-E. In addition, in each of the aforementioned cases, our results indicate whether players have either an expected payoff of zero, or positive expected payoffs, depending on their types.

In our model a player type with the higher (lower) value for the first prize will be referred to as an S-type player (W-type player, respectively). Our equilibrium analysis shows that, depending on the players' values for the prizes, either the W-type player(s) or the S-type player(s) has a positive expected payoff, but both types never have positive expected payoffs at the same time. Furthermore, if the S-type player is the asymmetric player, he is the only one who has a positive expected payoff. On the other hand, if the S-type players are the symmetric players, the asymmetric W-type player does not necessarily have an expected payoff of zero. In that case, depending on his value for the second prize, he might be the only player with a positive expected payoff although he is allegedly considered the weaker player. The intuitive explanation for such a scenario is that the

[2] Baye et al. (1996) showed that even in the all-pay auction with a single prize the players' strategies are not necessarily convex.

S-type players who have low values of the second prize are only interested in the first prize, though being aware that if they do not win the first prize they will end up with almost nothing. On the other hand, the W-type player knows that if he does not win the first prize he has a chance to win the second one which is also highly valuable for him. Therefore he fiercely competes against the S-type players and becomes the only player with a positive expected payoff. Hence, we conclude that although the values for the first (larger) prize have the greatest effect on the identity of the players with positive expected payoffs, the value of the second prize might have a non-negligible effect. In other words, the order of the players according to their expected payoffs depends on the valuations of all the prizes.

We then consider the all-pay auction with $n > 3$ players. Although we do not provide a complete analysis of this setup, we do show how our results for three players can be generalized. We prove that the asymmetric player may have a positive expected payoff, whether or not he has the higher value for the first prize. On the other hand, the $n - 1$ symmetric players will always have an expected payoff of zero. This is due to the fact that even if these players have higher values for either the first prize or for both prizes, the competition among them yields an expected payoff of zero.

We are not the first to deal with the all-pay auction with heterogeneous prizes. Incomplete-information auctions, where only the common distribution of private values is commonly known, has been studied, among others, by Moldovanu and Sela (2001, 2006), Moldovanu et al. (2012), and Liu and Lu (2017). In addition, complete-information auctions with identical prizes and linear costs in which the players' values are common knowledge has been studied by Barut and Kovenock (1998), and Clark and Riis (1998). Siegel (2010) analyzed such contests with nonlinear costs. Bulow and Levin (2006) studied all-pay auctions with heterogenous prizes and linear costs in which the first-order differences in successive prizes are constants, and Gonzalez-Diaz and Siegel (2013) extended this work by allowing nonlinear costs. Later, Xiao (2016) investigated another version of the all-pay auction with heterogenous prizes in which either the ratio of successive prizes is constant or the second-order differences are a positive constant.

The model most similar to ours, namely, having two symmetric players and one asymmetric player who compete over two prizes, was studied by Dahm (2018). However, this work places several restrictions on the prizes' values so that the value of the second prize is zero for the asymmetric player. Thus, Dahm is mainly interested in one prize, and considered the symmetric players' values for the first prize to be larger than the respective asymmetric player's value. Xiao (2018) also studied all-pay auctions with two nonidentical prizes, but he assumed that the sequence of prizes is either convex or concave, that is, the second-order differences (among prizes) are either a positive or a negative constant. Therefore, in these studies the heterogeneity among the prizes is limited by some special properties imposed on the sequence of the prizes' valuations.[3] Furthermore, it is assumed that the ratio of the values for every pair of prizes is the same for all the players who differ from each other by their ability or, alternatively, their bid cost. In other words, the players technically have the same value for each prize, but due to the heterogenous cost functions, they

[3] In Olzewski and Siegel (2016) the heterogeneity of the prizes is not limited, but they assume that the numbers of prizes and players go to infinity.

differ in their expected payoff for winning. Nevertheless, the ratio between the values of each pair of prizes is identical among all the players.

In contrast, in our model the players differ in their prize valuations and in the ratios among these valuations. This difference is crucial, since it eliminates the possibility to rank players according to their valuations for the first prize or, alternatively, by their marginal cost of effort, as in previous studies where the players' valuations for the second prize do not play a role. Thus, one of our main contribution in this work is to show that the players' valuations for the second prize are not negligible and may play a key role in determining the equilibrium and expected payoffs. Indeed, we show that in the three-player contest with two S-type players, who have relatively low values for the second prize, a W-type player with a relatively high value for the second prize may have the highest expected payoff.

The rest of the paper proceeds as follows. In Section 2, we introduce the model, and in Section 3, we present general properties of the equilibria. In Sections 4 and 5, we analyze the equilibrium strategies with three players and two heterogenous prizes where the players' supports are convex. In Section 6, we illustrate an equilibrium with non-convex supports, and generalize our equilibrium analysis to contests with more than three players. Section 7 concludes. Most of the proofs appear in the Appendix.

2 The model

We first consider a two-prize all-pay auction with three players. There are two types of players who differ in their prize valuations: the 'strong' type, denoted by S, who has valuations s_1 and s_2 for the first and second prize respectively, and the 'weak' type, denoted by W, who has valuations w_1 and w_2 for the first and second prizes. Note that $s_1 > s_2 \geqslant 0$ and $w_1 > w_2 \geqslant 0$. We refer to the types as strong and weak since $s_1 > w_1$ is the basic assumption that affects the type which has a positive expected payoff in equilibrium. Unless stated otherwise, we assume that among the three players, there are two S-type players and one W-type player.

The bid set of each player is $\mathbb{R}_+$ and, without loss of generality, we can assume that the bids of the S-type and W-type players are bounded on $[0, s_1]$ and $[0, w_1]$, respectively. A strategy of a player is a distribution over the set of feasible bids (i.e., the CDF) which is denoted by F_T for every $T \in \{S, W\}$. We denote the random bids of the S-type and W-type players by $X_S \in I_S$ and $X_W \in I_W$, where I_S and I_W are the relevant supports. The analysis is confined to symmetric equilibria with respect to the players' types. We assume that the player with the highest bid wins the first prize, and the player with the second highest bid wins the second prize.

Under the mentioned assumptions and given a strategy profile (F_S, F_W), the expected payoffs of both types under a bid of $x \in \mathbb{R}$ are

$$\begin{aligned} U_S(x|F_S, F_W) &= s_1 F_S(x) F_W(x) + s_2[F_W(x)(1 - F_S(x)) + F_S(x)(1 - F_W(x))] - x \qquad (1) \\ &= [(s_1 - 2s_2)F_S(x) + s_2]\, F_W(x) + s_2 F_S(x) - x \end{aligned}$$

and

$$\begin{aligned} U_W(x|F_S,F_W) &= w_1F_S^2(x)+2w_2F_S(x)(1-F_S(x))-x \\ &= (w_1-2w_2)F_S^2(x)+2w_2F_S(x)-x \end{aligned} \tag{2}$$

Note that the expected payoffs do not account for possible ties since ties occur with 0-probability in equilibrium.

3 General properties of equilibria

We first introduce some general properties of the equilibrium profile (F_S, F_W) when there are several S-type players and one W-type player. Unless otherwise stated, all proofs appear in the Appendix.

Lemma 1 *In a symmetric equilibrium, F_S has no atoms in $[0, s_1]$ and F_W has no atoms in $(0, w_1]$.*

Lemma 1 proves that the only possible atom in a symmetric equilibrium is for the W-type player to bid 0 with a strictly positive probability. Otherwise, all other bids by all types are individually chosen with 0-probability. Following Lemma 1, we can deduce the following corollary.

Corollary 1 *In a symmetric equilibrium, if $\Pr(X_W \in [0,\epsilon)) > 0$ for any $\epsilon > 0$, $U_W(x|F_S, F_W) = 0$ for any $x \in I_W$.*

The proof is omitted since it is a straightforward conclusion from the fact that F_S is non-atomic at 0. Namely, since the payoffs are right-side continuous and without an atom at 0 of an S-type player, then the point-wise expected payoff of the W-type player converges to zero when a bid x approaches 0. Therefore, by the indifference principle, the expected payoff must be zero.

The next lemma depicts how the random equilibrium bids of the different types relate to one another, in terms of their supports.

Lemma 2 *In a symmetric equilibrium, for every open interval $I \subseteq \mathbb{R}_{++}$ such that $\Pr(X_W \in I) > 0$, it follows that $\Pr(X_S \in I) > 0$.*

Lemma 2 suggests that for every symmetric equilibrium in which the random bids X_S of the S-type players and the random bid X_W of the W-type player are supported on I_S and I_W, respectively, then $I_W \subseteq I_S$ up to a zero-measure deviation of the S-type players. We can now use these results to characterize two key properties of the S-type players equilibrium bids: (i) in Lemma 3 we prove that the support of the S-type players' is a connected set; and (ii) in Lemma 4 we extend this result by showing that the their minimal bid is zero, in every symmetric equilibrium.

Lemma 3 *In a symmetric equilibrium, I_S is a connected set.*

Lemma 4 *In a symmetric equilibrium, $\inf I_S = 0$.*

Lemmas 1–4 enable us to prove that, in every symmetric equilibrium, there exists at least one type with an expected payoff of zero. This result is given in the following Lemma 5.

Lemma 5 *In a symmetric equilibrium, either* $U_W(x|F_S, F_W) = 0$ *for any* $x \in I_W$*, or* $U_S(x|F_S, F_W) = 0$ *for any* $x \in I_S$*.*

The proof of Lemma 5 also shows that, in every symmetric equilibrium, if the W-type player has no atom at 0, then the expected payoffs of the S-type players are zero. To see this, note that the proof of Lemma 5 directly relates to the case in which the support I_W of the W-type player's bids is strictly above 0. Otherwise, one can assume that $\inf I_W = 0$ while $\Pr(X_W = 0) = 0$. Again, the fact that $\inf I_S = 0$ implies that a bid of 0 yields no positive expected payoff because all other bids are strictly above 0 with probability 1. We can now follow the same construction as in the proof of Lemma 5, using continuity and the indifference principle, to establish that $U_S(x|F_S, F_W) = 0$ for every $x \in I_S$. Let us formalize this in the following corollary.

Corollary 2 *In a symmetric equilibrium, if* $\Pr(X_W = 0) = 0$*, then* $U_S(x|F_S, F_W) = 0$ *for any* $x \in I_S$*.*

It is important to note that these results facilitate the identification of equilibria, as a function of the model's different parameters. Moreover, they also assist us to classify important properties of several equilibria in the following sections (see Cases C and D below).

4 Three-player contests with one weak and two strong players

We next analyze the equilibrium in the all-pay auction with three players who compete for two heterogeneous prizes. We assume that there are two S-type players and one W-type player. Below, we divide our analysis into four cases, A-D, depending on the players' valuations of the prizes. Specifically,

- Case A provides an equilibrium under the condition that $s_1 - s_2 \geqslant \max\{w_1, 2w_2\}$;
- Case B provides an equilibrium under the condition that $w_1 \geqslant s_1 - s_2$;
- Cases C and D conclude the analysis under the condition that $2w_2 > s_1 - s_2 \geqslant w_1$.

Before we proceed with each of these cases, let us provide one more general property of the equilibria below, given that $2w_2 > w_1$. This is obviously relevant for Cases C and D, as classified above, but could also apply to the other cases as well. The following result shows that in case $2w_2 > w_1$, then a symmetric equilibrium dictates that the support of the W-type player is also a connected set.

Lemma 6 *In a symmetric equilibrium, if* $2w_2 > w_1$*, then* I_W *is a connected set.*

The result given in Lemma 6 enables us to classify the equilibria in Cases C and D more accurately, knowing that the supports of all players' bids are convex.

4.1 Case A: The weak player stays out of the contest.

Here the W-type player stays out of the contest, and the two S-type players compete against each other, so that each wins one of the prizes.

Proposition 1 *In the all-pay auction with two S-type players and one W-type player, if $[s_1 - s_2] \geqslant \max\{w_1, 2w_2\}$, there exists an equilibrium in which both S-type players randomize on the interval $[0, s_1 - s_2]$ according to their cumulative distribution bid function $F_S(x)$ which is*

$$F_S(x) = \begin{cases} 0, & \text{for } x < 0, \\ \frac{x}{s_1 - s_2}, & \text{for } 0 \leqslant x \leqslant s_1 - s_2, \\ 1, & \text{for } x \geqslant s_1 - s_2, \end{cases} \tag{3}$$

while the W-type player bids 0 with probability 1. Under this equilibrium, the expected payoff of both S-type players is s_2, while the expected payoff of the W-type player is 0.

The intuition behind this result is quite clear. Since $s_1 - s_2 \geqslant \max\{w_1, 2w_2\}$ the two S-type players are too strong relative to the W-type player such that he stays out of the contest, and the two S-type players compete against each other. Their expected payoff is their minimal possible value for the second prize s_2.

The following example illustrates an equilibrium under the conditions of Proposition 1.

Example 1 *Assume that there are two S-type players whose prize valuations are $s_1 = 10$, $s_2 = 5$, and a W-type player whose prizes' valuations are $w_1 = 4$, $w_2 = 2$, so that the conditions of* Proposition 1 *hold. Then, the mixed-strategy equilibrium described in* Proposition 1 (*see* Figure 1) *is*

$$F_S(x) = \begin{cases} 0, & \text{for } x < 0, \\ \frac{x}{5}, & \text{for } 0 \leqslant x \leqslant 5, \\ 1, & \text{for } x \geqslant 5, \end{cases} \qquad F_W(x) = \begin{cases} 0, & \text{for } x < 0, \\ 1, & \text{for } x \geqslant 0. \end{cases}$$

The expected payoff of each S-type player is 5, and that of the W-type player is 0.

If the conditions of Proposition 1 are violated, the W-type player may actually compete. Therefore, we continue our analysis by studying the equilibria where all players compete against each other (i.e., support a strictly positive bid with probability 1), and both types, S and W, use mixed strategies with a common support. Since the players' support in equilibrium varies with their valuations (s_1, s_2, w_1, w_2), our analysis is divided into cases in which each one relates to a different equilibrium structure. Specifically, Case A relates to situations where $s_1 - s_2 \geqslant \max\{w_1, 2w_2\}$. Accordingly, Case B relates to situations where $w_1 \geqslant s_1 - s_2$, whereas Cases C and D complete the analysis with equilibria when $2w_2 > s_1 - s_2 \geqslant w_1$.

4.2 Case B: All the players have symmetric supports

The following proposition concerns cases where $w_1 \geqslant s_1 - s_2$. It derives an equilibrium where all players participate in the contest with probability 1, and where heir common support is $[0, w_1]$.

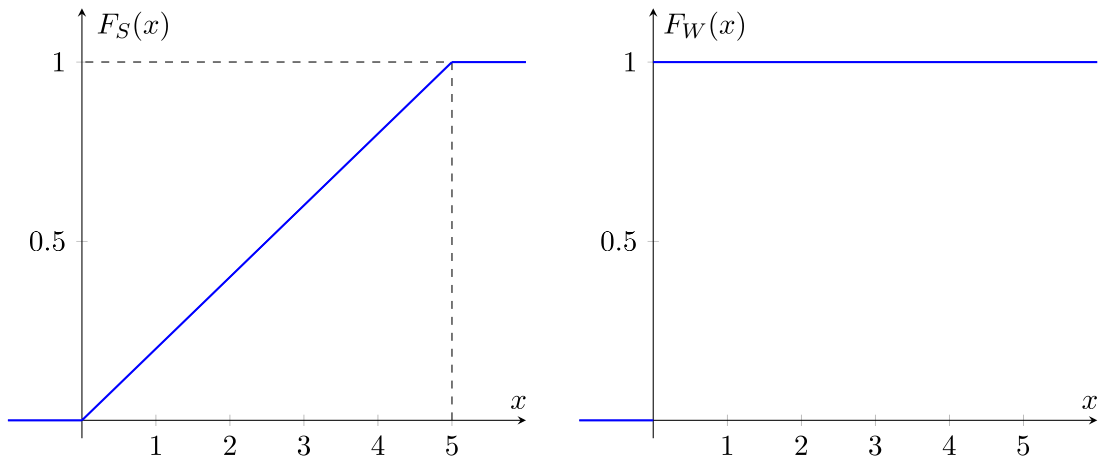

Figure 1: The distributions of the S- and W-type players, in equilibrium, given $s_1 = 10, s_2 = 5, w_1 = 4$, and $w_2 = 2$ (these values meet the conditions of Proposition 1).

Proposition 2 *In the all-pay auction with two S-type players and one W-type player, assume that $w_1 \geqslant s_1 - s_2$ and consider the following functions*

$$F_S(x) = \begin{cases} 0, & \text{for } x < 0, \\ \frac{w_2 - \sqrt{w_2^2 - 2w_2 x + w_1 x}}{2w_2 - w_1}, & \text{for } 0 \leqslant x \leqslant w_1, \\ 1, & \text{for } x > w_1, \end{cases} \qquad F_W(x) = \begin{cases} 0, & \text{for } x < 0, \\ \frac{x - s_2 F_S(x) + s_1 - w_1}{(s_1 - 2s_2) F_S(x) + s_2}, & \text{for } 0 \leqslant x \leqslant w_1, \\ 1, & \text{for } x > w_1. \end{cases} \tag{4}$$

If F_W is non-decreasing, then the strategy profile (F_S, F_W) is an equilibrium in which all the players randomize on the interval $[0, w_1]$. Under this equilibrium, the expected payoff of both S-type players is $s_1 - w_1$, whereas the expected payoff of the W-type player is 0.

The intuition behind this result is that since $s_1 > w_1$ and $w_1 \geqslant s_1 - s_2$ the two S-type players are stronger than the W-type player, but they are not too strong such that all the players compete. Namely, the difference $s_1 - s_2$ is the marginal gain for an S-type player competing over the first prize, rather than the second prize. Since $w_1 > s_1 - s_2$, the W-type player as a greater incentive to compete for *his* first-prize, relative to the marginal gain of two S-type players competing against each other over the two prizes. Thus, the W-type actively competes in the contest. Yet, the expected payoff of the two S-type players, who remain the stronger ones in the contest, is the highest value of the W-type player. On the other hand, the W-type player who is the weakest one has an expected payoff of 0. In other words, the S-type players derive their rents only to the highest value that the W-type player can compete, i.e., w_1.

The following example shows that the conditions of Proposition 2 are feasible, and that there are parameters which simultaneously support the required constraints.

Example 2 *Assume that there are two S-type players whose prize valuations are $s_1 = 10, s_2 = 6$, and a W-type player whose prize valuations are $w_1 = 8$ and $w_2 = 3$, so that the conditions of*

Proposition 2 *hold. Then, a mixed-strategy equilibrium* (*see* Figure 2) *is*

$$F_S(x) = \begin{cases} 0, & \text{for } x < 0, \\ \frac{\sqrt{9+2x}-3}{2}, & \text{for } 0 \leqslant x \leqslant 8, \\ 1, & \text{for } x > 8, \end{cases} \qquad F_W(x) = \begin{cases} 0, & \text{for } x < 0, \\ \frac{x-3\sqrt{9+2x}+11}{9-\sqrt{9+2x}}, & \text{for } 0 \leqslant x \leqslant 8, \\ 1, & \text{for } x > 8. \end{cases}$$

The expected payoff of each S-type player is 2*, while the expected payoff of the W-type player is* 0.

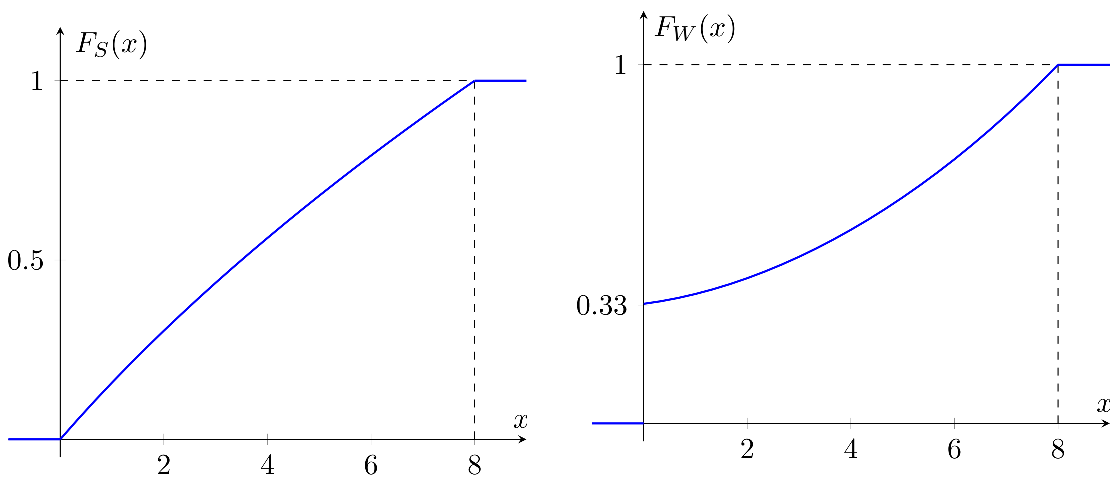

Figure 2: The distributions of the S- and W-type players, in equilibrium, given $s_1 = 10, s_2 = 6, w_1 = 8$, and $w_2 = 3$. Note that these parameters meet the conditions of Proposition 2.

4.3 Case C: The weak player has a one-sided short support

In Case C, all the players participate and none of them stays out with a positive probability. However, the W-type player has a shorter support relative to the S-type players. Namely, the W-type player's maximal bid is smaller than the maximal bids of the S-type players.

Proposition 3 *In the all-pay auction with two S-type players and one W-type player, assume that* $2w_2 > s_1 - s_2 \geqslant w_1$, $K_1 = s_2 - \frac{[2w_2-(s_1-s_2)]^2}{4(2w_2-w_1)} \geqslant 0$, *and consider the following functions*

$$F_S(x) = \begin{cases} 0, & \text{for } x < 0, \\ \frac{w_2-\sqrt{w_2^2-2w_2x+w_1x}}{2w_2-w_1}, & \text{for } 0 \leqslant x \leqslant \alpha, \\ 1+\frac{x+K_1-s_1}{s_1-s_2}, & \text{for } \alpha \leqslant x \leqslant s_1-K_1, \\ 1, & \text{for } x > s_1-K_1, \end{cases} \qquad F_W(x) = \begin{cases} 0, & \text{for } x < 0, \\ \frac{x-s_2F_S(x)+K_1}{(s_1-2s_2)F_S(x)+s_2}, & \text{for } 0 \leqslant x \leqslant \alpha, \\ 1, & \text{for } x > \alpha, \end{cases} \tag{5}$$

where $\alpha = \frac{(2w_2)^2-(s_1-s_2)^2}{4(2w_2-w_1)}$. *If F_W is non-decreasing, then the strategy profile* (F_S, F_W) *is an equilibrium in which the W-type player randomizes on the interval* $[0, \alpha]$ *according to F_W, and the S-type players randomize on the interval* $[0, s_1 - K_1]$ *according to F_S. Under this equilibrium, the expected payoff of both S-type players is K_1, and that of the W-type player is* 0.

The intuition behind this result is that the two S-type players are stronger than the W-type player, but they are less strong than in case B since the W-type player's value for the second prize here is larger and also plays a role. Specifically, the fact that $s_1 - s_2 \geqslant w_1$, limits the ability of the W-type player to compete over the first prize (see the discussion following Proposition 2), whereas the condition $2w_2 > s_1 - s_2$ pushes this player to compete over the second prize, so that the W-type player's maximal bid falls short of the S-type players' maximal one. In return, the expected payoff of the two S-type players who are the stronger ones is smaller than the highest value of the W-type player. The expected payoff of the W-type player who is the weakest one is 0.

The following example shows that the conditions of Proposition 3 are feasible, and that there are parameters which simultaneously support all the necessary constraints.

Example 3 *Assume that there are two S-type players whose prize valuations are $s_1 = 10, s_2 = 4$ and a W-type player whose prize valuations are $w_1 = 5$, and $w_2 = 4$, so that the conditions of* Proposition 3 *hold. Then, the mixed-strategy equilibrium described in* Proposition 3 (*see* Figure 3) *is*

$$F_S(x) = \begin{cases} 0, & \text{for } x < 0, \\ \frac{4-\sqrt{16-3x}}{3}, & \text{for } 0 \leqslant x \leqslant 7/3, \\ \frac{3x-1}{18}, & \text{for } 7/3 \leqslant x \leqslant 19/3, \\ 1, & \text{for } x > 19/3, \end{cases} \qquad F_W(x) = \begin{cases} 0, & \text{for } x < 0, \\ \frac{3x+4\sqrt{16-3x}-5}{20-2\sqrt{16-3x}}, & \text{for } 0 \leqslant x \leqslant 7/3, \\ 1, & \text{for } x > 7/3. \end{cases}$$

The expected payoff of each S-type player is $6\frac{1}{3}$, while the expected payoff of the W-type player is 0.

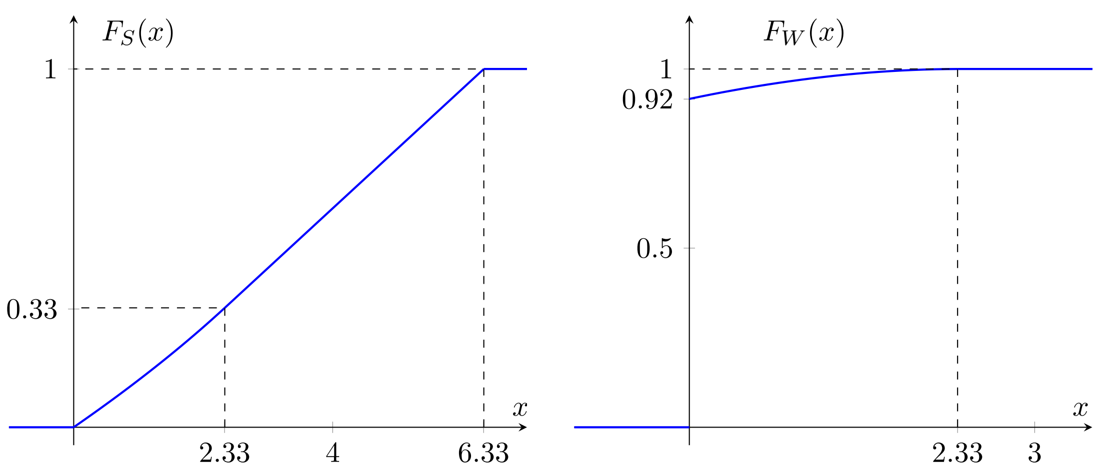

Figure 3: The distributions of the S- and W-type players, in equilibrium, given $s_1 = 10, s_2 = 4, w_1 = 5$, and $w_2 = 4$. These values meet the conditions of Proposition 3.

4.4 Case D: The weaker player has a two-sided short support

In this case, both types support a positive bid with a probability of 1, but the W-type player has a shorter support relative to the S-type players. Specifically, the W-type player's maximal bid is smaller than that of the S-type players, and the W-type player's minimal bid is larger than that of the S-type players.

Proposition 4 *In the all-pay auction with two S-type players and one W-type player, assume that $2w_2 > s_1 - s_2 \geqslant w_1$, $K_1 = s_2 - \frac{[2w_2-(s_1-s_2)]^2}{4(2w_2-w_1)} < 0$, and consider the following functions*

$$F_S(x) = \begin{cases} 0, & \text{for } x < 0, \\ \frac{x}{s_2}, & \text{for } 0 \leqslant x \leqslant \alpha_1, \\ \frac{w_2-\sqrt{w_2^2-(x-K_1)(2w_2-w_1)}}{2w_2-w_1}, & \text{for } \alpha_1 \leqslant x \leqslant \alpha_2, \\ \frac{x-s_2}{s_1-s_2}, & \text{for } \alpha_2 \leqslant x \leqslant s_1, \\ 1, & \text{for } x > s_1, \end{cases} \quad F_W(x) = \begin{cases} 0, & \text{for } x < \alpha_1, \\ \frac{x-s_2F_S(x)}{(s_1-2s_2)F_S(x)+s_2}, & \text{for } \alpha_1 \leqslant x \leqslant \alpha_2, \\ 1, & \text{for } x > \alpha_2, \end{cases} \tag{6}$$

where

$$\alpha_1 = s_2\frac{2w_2-s_2-\sqrt{(2w_2-s_2)^2+4K_1(2w_2-w_1)}}{2(2w_2-w_1)} \quad \text{and} \quad \alpha_2 = s_2 + (s_1-s_2)\frac{2w_2-(s_1-s_2)}{2(2w_2-w_1)}.$$

If F_W is non-decreasing, then the strategy profile (F_S, F_W) is an equilibrium in which the S-type players randomize on the interval $[0, s_1]$ according to F_S, and the W-type player randomizes on the interval $[\alpha_1, \alpha_2]$ according to F_W. Under this equilibrium, the expected payoff of the S-type players is 0, while the expected payoff of the W-type player is $-K_1$.

The intuition behind this result is that the W-type player's value for the second prize is much larger than that of the S-type players such that he has an incentive to compete for both of the prizes, while the S-type players actually compete for the first one only. The fact that there are two S-type players who compete over the single (first) prize eliminates their ability to derive positive expected payoffs. Thus, the W-type player is the only player with a positive expected payoff, and the S-type players have an expected payoff of 0.

The following example illustrates that the conditions of Proposition 4 are feasible, and that there are parameters that simultaneously support all the required constraints.

Example 4 *Assume that there are two S-type players whose prize valuations are $s_1 = 30, s_2 = 1$, and a W-type player whose prize valuations are $w_1 = 25$, and $w_2 = 20$, so that the conditions of* Proposition 4 *hold. Then, the players' mixed-strategy equilibrium-strategies (see* Figure 4*) are*

$$F_S(x) = \begin{cases} 0, & \text{for } x < 0, \\ x, & \text{for } 0 \leqslant x \leqslant \frac{39-\sqrt{1460}}{30} \approxeq 0.026, \\ \frac{40-\sqrt{1539-60x}}{30}, & \text{for } 0.026 \approxeq \frac{39-\sqrt{1460}}{30} \leqslant x \leqslant \frac{349}{30} \approxeq 11.633, \\ \frac{x-1}{29}, & \text{for } 11.633 \approxeq \frac{349}{30} \leqslant x \leqslant 30, \\ 1, & \text{for } x > 30, \end{cases}$$

$$F_W(x) = \begin{cases} 0, & \text{for } x < \frac{39-\sqrt{1460}}{30} \approxeq 0.026, \\ \frac{30x-40+\sqrt{1539-60x}}{1150-28\sqrt{1539-60x}}, & \text{for } 0.026 \approxeq \frac{39-\sqrt{1460}}{30} \leqslant x \leqslant \frac{349}{30} \approxeq 11.633, \\ 1, & \text{for } x > \frac{349}{30} \approxeq 11.633. \end{cases}$$

The expected payoff of each S-type player is 0, and that of the W-type player is 1.2.

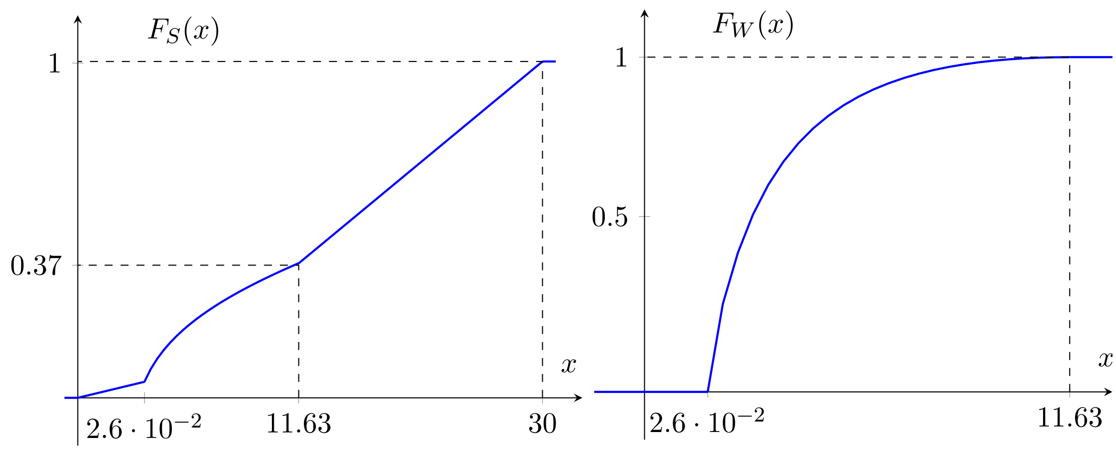

Figure 4: The distributions of the S- and W-type players, in equilibrium, given $s_1 = 30, s_2 = 1, w_1 = 25$, and $w_2 = 20$. The axis at the LHS of the distribution are scaled to reflect to differences between the two distributions. Note that these parameters meet the conditions of Proposition 4.

We can now use our basic results from Section 3 and Lemma 6 to prove that the equilibrium structure given in Proposition 4 is unique under the conditions of Case D. More specifically, assuming that $2w_2 > s_1 - s_2 \geqslant w_1$ and $s_2 - \frac{[2w_2-(s_1-s_2)]^2}{4(2w_2-w_1)} < 0$, Lemma 6 along with previous results indicate that the supports of all players' bids are convex (i.e., intervals). Thus, the other possible equilibrium structure is an equilibrium in which the W-type player maintains an atom at $x = 0$. The following lemma shows this is not possible.

Lemma 7 *Consider an all-pay auction with two S-type players and one W-type player, and assume that $2w_2 > s_1 - s_2 \geqslant w_1$, $s_2 - \frac{[2w_2-(s_1-s_2)]^2}{4(2w_2-w_1)} < 0$. Then, in a symmetric equilibrium (F_S, F_W) it must be that $F_W(0) = 0$.*

In simple terms, the lemma states that there exists no symmetric equilibrium under which the W-type player maintains an atom at $x = 0$. Thus, the equilibrium structure given in Proposition 4 is somewhat unique, under the given parametric assumptions.

5 Three-player contests with one strong and two weak players

In this section, we assume that there are two W-type players and a single S-type player. Thus, given a strategy profile (F_S, F_W), the expected payoffs of all he types under a bid of $x \in \mathbb{R}$ are

$$\begin{aligned} U_W(x|F_S, F_W) &= w_1 F_W(x) F_S(x) + w_2[F_S(x)(1 - F_W(x)) + F_W(x)(1 - F_S(x))] - x \\ &= [(w_1 - 2w_2)F_W(x) + w_2] F_S(x) + w_2 F_W(x) - x, \end{aligned}$$

and

$$\begin{aligned} U_S(x|F_S, F_W) &= s_1 F_W^2(x) + 2s_2 F_W(x)(1 - F_W(x)) - x \\ &= (s_1 - 2s_2)F_W^2(x) + 2s_2 F_W(x) - x. \end{aligned}$$

5.1 Case E: The strong player has a one-sided short support

In this set-up of two W-type players and a single S-type player, our equilibrium analysis shows that both types participate with a probability of 1, but the S-type player has a shorter support relative to the W-type players. Specifically, the S-type player's minimal bid is larger than that of the W-type players.

Proposition 5 *In the all-pay auction with two W-type players and one S-type player, assume that*

$$0 < \alpha = \frac{w_2}{2\Delta(s)} \left[-2s_2 + w_2 + \sqrt{(2s_2 - w_2)^2 + 4\Delta(s)(s_1 - w_1)} \right] < w_1 \quad , \text{ where } \Delta(s) = s_1 - 2s_2,$$

and consider the following functions

$$F_W(x) = \begin{cases} 0, & \text{for } x < 0, \\ \frac{x}{w_2}, & \text{for } 0 \leqslant x \leqslant \alpha, \\ \frac{-s_2 + \sqrt{s_2^2 + \Delta(s)(s_1 - w_1 + x)}}{\Delta(s)}, & \text{for } \alpha \leqslant x \leqslant w_1, \\ 1, & \text{for } x > w_1, \end{cases} \quad F_S(x) = \begin{cases} 0, & \text{for } x < \alpha, \\ \frac{x - w_2 F_W(x)}{(w_1 - 2w_2)F_W(x) + w_2}, & \text{for } \alpha \leqslant x \leqslant w_1, \\ 1, & \text{for } x > w_1. \end{cases} \tag{7}$$

If F_S is non-decreasing, then the strategy profile (F_S, F_W) is an equilibrium in which the W-type players randomize on the interval $[0, w_1]$ according to F_W, and the S-type player randomizes on the interval $[\alpha, w_1]$ according to F_S. Under this equilibrium, the expected payoff of both W-type players is 0, and that of the S-type player is $s_1 - w_1$.

The intuition behind this result is quite clear since the S-type player's value for the first prize is larger than that of the W-type players such that the S-type player is the strong one. Therefore he is the only one who has a positive expected payoff which is equal to the difference between the S-type player's value of the first prize and the W-type players' value.

The following example shows that the conditions of Proposition 5 are feasible, and that there are parameters which simultaneously support all the required constraints.

Example 5 *Assume that there is a single S-type player whose prize valuations are $s_1 = 5, s_2 = 2$, and two W-type players whose prize valuations are $w_1 = 3$, and $w_2 = 2$, so that the conditions of* Proposition 5 *hold, where*

$$\alpha = \tfrac{w_2}{2\Delta(s)}\left[-2s_2 + w_2 + \sqrt{(2s_2 - w_2)^2 + 4\Delta(s)(s_1 - w_1)}\right] = \sqrt{12} - 2.$$

Then, the players' mixed-strategy equilibrium-strategies described in Proposition 5 (*see* Figure 5) *are*

$$F_W(x) = \begin{cases} 0, & \text{for } x < 0, \\ \frac{x}{2}, & \text{for } 0 \leqslant x \leqslant \sqrt{12} - 2, \\ -2 + \sqrt{6 + x}, & \text{for } \sqrt{12} - 2 \leqslant x \leqslant 3, \\ 1, & \text{for } x > 3, \end{cases} \quad F_S(x) = \begin{cases} 0, & \text{for } x < \sqrt{12} - 2, \\ \frac{x+4-2\sqrt{6+x}}{4-\sqrt{6+x}}, & \text{for } \sqrt{12} - 2 \leqslant x \leqslant 3, \\ 1, & \text{for } x > 3. \end{cases}$$

The expected payoff of the S-type player is 2, *while that of each of the W-type players is* 0.

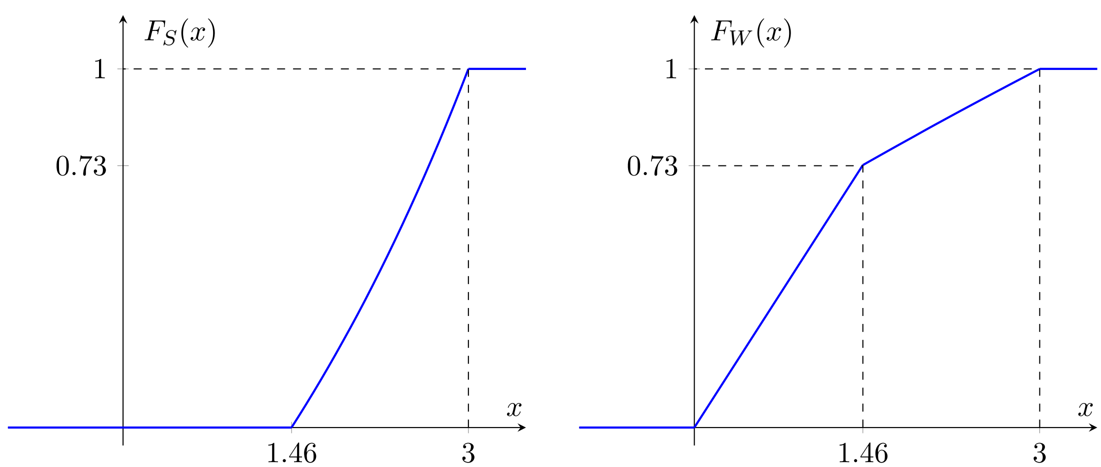

Figure 5: The distributions of the S- and W-type players, in equilibrium, given $s_1 = 5, s_2 = 2, w_1 = 3$, and $w_2 = 2$. These values sustain the conditions of Proposition 5.

6 Extensions

6.1 A non-convex support for the W-type player

In Cases $B - E$, we derived the equilibrium structure given that the stated functions F_W is non-decreasing. However, under some parametric conditions, this assumption does not hold and a different type of equilibrium arises. Specifically, if we consider the distribution F_W given in Proposition 2, one can fix the valuations such that F_W may decrease close to zero.[4] Thus, we need to derive new equilibrium strategies for which the support of the W-type player's strategy is non-convex. The following statement provides an equilibrium structure of this kind.

[4] An easy way to see this is by substituting $w_2 = 3$ with $w_2' = 0$ in Example 2.

Claim 1 *Assume that there are two S-type players whose values of the prizes are $s_1 = 8, s_2 = 6$, and a single W-type player whose values are $w_1 = 4$, and $w_2 = 0$. Then, a mixed-strategy equilibrium* (*see* Figure 6) *is*

$$F_S(x) = \begin{cases} 0, & \text{for } x < 0, \\ \frac{3x}{10}, & \text{for } 0 \leqslant x \leqslant 25/9 \\ \frac{\sqrt{x}}{2}, & \text{for } 25/9 \leqslant x \leqslant 4 \\ 1, & \text{for } x > 4, \end{cases} \qquad F_W(x) = \begin{cases} 0, & \text{for } x < 0, \\ \frac{2}{3}, & \text{for } 0 \leqslant x \leqslant 25/9, \\ \frac{4+x-3\sqrt{x}}{6-2\sqrt{x}}, & \text{for } 25/9 \leqslant x \leqslant 4 \\ 1, & \text{for } x > 4. \end{cases} \tag{8}$$

Note that F_W is not strictly increasing, and is fixed for all $0 \leqslant x \leqslant 25/9$. In that case, the expected payoffs of both S-type players is 4, and that of the W-type player is 0.

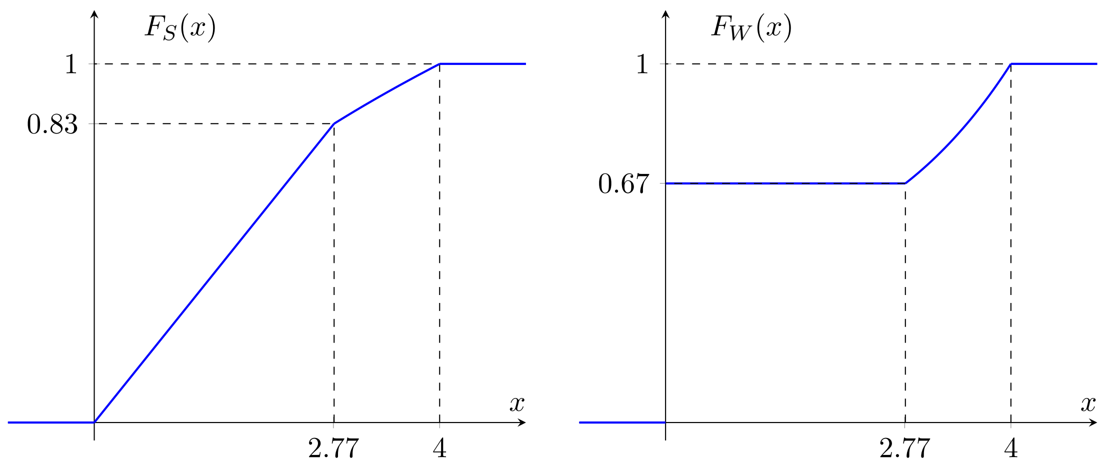

Figure 6: The distributions of the S- and W-type players, in equilibrium, given $s_1 = 8, s_2 = 6, w_1 = 4$, and $w_2 = 0$ (as given in Claim 1).

Note that this example perfectly fits the result of Lemma 6, because the condition $2w_2 > w_1$ does not hold, and the support of the W-type player need not be connected. One can even show, similarly to Lemma 6, that it is better for the W-type player to randomize on end points of a given interval, unless F_S is sufficiently concave (rather than linear). which is indeed the case in Claim 1.

6.2 More than three players

We now proceed to study the case of $n > 3$ players where there are at least three players of the same type and one player of a different type. This model is not only tractable, but even simpler to analyze than the three-player contest, since the competition among more than two players of the same type, regardless of whether their type is S or W, implies that their expected payoffs are zero. This is demonstrated in the following propositions, where in Proposition 6 there are multiple S-type players, and in Proposition 7 there are multiple W-type players.

Proposition 6 *In the all-pay auction with $n-1$ S-type players and one W-type player, where either $[s_1 - (n-2)s_2] \geqslant \max\{w_1, (n-1)w_2\}$ or $(n-2)s_2 \geqslant (n-1)w_2$ hold, there exists an equilibrium* *where the S-type players randomize on the interval $[0, s_1]$ according to their cumulative distribution bid function $F_S(x)$ which is given by*

$$s_1 F_S^{n-2}(x) + s_2(n-2)F_S^{n-3}(x)[1 - F_S(x)] - x = 0, \tag{9}$$

while the single W-type player bids 0 with a probability of 1. Then, the expected payoffs of all the players is 0.

The intuition behind this result is that the number of the S-type players who are the strong players $(n-1)$ is larger than the the number of the prizes (2). Then, as in the standard all-pay auction with only one prize, if the number of players with the highest value of the prize is larger than one, the competition is so intensive that the players' expected payoff is zero. Similarly, here the expected payoff of all the players is zero as well.

Now, let us consider the case with more-than-two W-type players and a single S-type one.

Proposition 7 *Consider an all-pay auction with $n-1$ W-type players, one S-type player, and the functions*

$$F_W(x) = \begin{cases} 0, & \text{for } x < 0, \\ \left[\frac{x}{w_2}\right]^{\frac{1}{(n-2)}}, & \text{for } 0 \leqslant x \leqslant \alpha_1, \\ G(x), & \text{for } \alpha_1 \leqslant x \leqslant w_1, \\ 1, & \text{for } x > w_1, \end{cases} \quad F_S(x) = \begin{cases} 0, & \text{for } x < \alpha_1, \\ \frac{x - w_2 F_W^{n-2}(x)}{F_W^{n-3}(x)[(w_1-(n-2)w_2)F_W(x)+w_2(n-3)]}, & \text{for } \alpha_1 \leqslant x \leqslant w_1, \\ 1, & \text{for } x > w_1, \end{cases} \tag{10}$$

where α_1 and $G(x)$ are given by

$$\begin{aligned} s_1 - w_1 + \alpha_1 &= s_1 \left[\frac{\alpha_1}{w_2}\right]^{(n-1)/(n-2)} + s_2(n-2)\frac{\alpha_1}{w_2}\left[1 - \left[\frac{\alpha_1}{w_2}\right]^{1/(n-2)}\right], \\ s_1 - w_1 + x &= s_1 G^{n-1}(x) + s_2(n-2)G^{n-2}(x)[1 - G(x)]. \end{aligned}$$

If $F_S(\cdot)$ is non-decreasing on $[\alpha_1, w_1]$ and $s_1 \geqslant s_2(n-2)$, then there exists an equilibrium in which the W-type players randomize on the interval $[0, w_1]$ and the S-type player randomizes on the interval $[a_1, w_1]$ according to the given strategies (F_S, F_W). Moreover, under this equilibrium, the expected payoff of all the W-type players are 0, while the expected payoff of the single S-type player is $s_1 - w_1$.

The intuition behind this result is clear and is similar to that of case E. Here, however, the S-type player's value for the first prize is larger than that of the $n-1$ W-type players such that the S-type player is the strong one. Therefore, he is the only one to have a positive expected payoff.

7 Conclusion

Most of the contest literature has focused on all-pay auctions with a single prize or several identical prizes. In the current work, we study all-pay auctions with heterogeneous prizes and demonstrate that the equilibrium strategies might be rather complex. In particular, we analyze the equilibrium strategies and show that the results may significantly differ from the standard all-pay auctions with either identical or heterogeneous prizes where the ratio of each pair of prizes is the same for all the players. We demonstrate that the identity of the dominant player, namely, the player with the highest expected payoff changes in a non-trivial manner, depending on the heterogeneity of the prizes. In other words, when some players have high values for the first prize, but low values for the second one, they do not necessarily have higher expected payoffs than players who have low values for the first prize but high values for the second prize. The reason is that the players who have high values for the second prize have some 'insurance' should they not win the first prize, while the other players compete for the first prize only, similarly to a "take-it-or-leave-it" competition. Thus, this paper sheds light on the key role of the second prize. Furthermore, the implications of our results is that in contests in which the number of prizes is smaller than the number of players, if some of the players have much higher values for the first prize than the others, the designer who wishes that for all the players to take part in the contest should be aware that the players with the low values for the first prize will have much higher values for the second prize. In that case, all the players will exert non-negligible efforts, the competition will be more intensive and more complex.

8 Appendix

8.1 Proof of Lemma 1

Proof. We begin with the CDF F_S. Assume, by contradiction, that there exists a symmetric equilibrium where all S-type players support some atom $a \in [0, s_1)$. There are at least two S-type players, so a tie occurs with positive probability, and a symmetric tie-breaking rule dictates a final allocation. Now consider an infinitesimal and unilateral upward-deviation of an S-type player, from a to $a + \epsilon < s_1$. On the one hand, this deviation increases the player's cost by an infinitesimal amount, but on the other, the expected prize increases by a strictly positive and relatively high amount due to the increased probability of winning without the need to split the prize according to some tie-breaking rule.[5] Thus, in a symmetric equilibrium, the bids' distributions of S-type players have no atoms in $[0, s_1)$.

Now assume, by contradiction, that there exists an atom $a = s_1$. In such a case, a tie occurs with positive probability, and a symmetric tie-breaking rule implies that in every such realization one S-type player receives s_1, whereas the other player receives $s_2 < s_1$. In other words, conditional on a tie at s_1, the expected payoff for both S-type players is negative, because both their bids equal s_1, whereas their expected reward is $\frac{s_1+s_2}{2} < s_1$. Thus, we conclude that such an atom cannot exist.

For the CDF F_W, we assume, by contradiction, that there exists a symmetric equilibrium in which the W-type player supports some atom $a \in (0, w_1]$. Since a cannot be an atom of X_S, either there exists some small $\epsilon > 0$ such that $\Pr(X_S \in (a - \epsilon, a)) > 0$, or there exists $\epsilon^* > 0$ such that $\Pr(X_S \in (a - \epsilon, a)) = 0$ for every $\epsilon \in (0, \epsilon^*)$. If the latter is the case, then the W-type player has

[5] The tie-breaking rule does not have to be symmetric, and any rule would motivate at least one player to shift the private bid upwards.

a profitable deviation downwards. Specifically, for some $\epsilon > 0$, bids in $(a - \epsilon, a)$ are not supported by the S-type players, so the W-type player can shift his atom from a to $a - \frac{\epsilon}{2}$ such that the probability of getting a prize is not affected while the cost decreases. If, however, there exists some small $\epsilon > 0$ such that $\Pr(X_S \in (a - \epsilon, a)) > 0$, then any of the S-type players can shift bids from this small interval upwards to $a + \epsilon'$, for some small $\epsilon' > 0$. Such a deviation increases the expected payoff by strictly increasing the probability of winning the first prize, while the increased cost is infinitesimal. Thus, we can conclude that this cannot be an equilibrium, and F_W has no interior atoms in equilibrium. ■

8.2 Proof of Lemma 2

Proof. Fix a symmetric equilibrium (F_S, F_W). Assume, by contradiction, that there is an open interval I such that $\Pr(X_W \in I) > 0 = \Pr(X_S \in I)$. If the W-type player has an atom $a \in I$, then there exists a strictly profitable deviation downwards from a to $a' \in (\inf I, a) \subset I$ since the probability of winning a prize does not change while the realized cost strictly decreases. Moreover, even if the W-type player has no atoms in I, then the player can shift a positive-probability set of values (from I) downwards in a similar manner to the atom shift, while remaining within I. Again, this would not change the probability of winning a prize, whereas the realized cost strictly decreases. Therefore, we conclude that this cannot be an equilibrium since the W-type player always has a strictly profitable deviation. ■

8.3 Proof of Lemma 3

Proof. Assume, by contradiction, that I_S is not a connected set. By the lack of interior atoms, there exists an open interval $I \subset \mathbb{R}_{++}$ such that $\Pr(X_S \geqslant \sup I) \cdot \Pr(X_S \leqslant \inf I) > 0 = \Pr(X_S \in I)$. By Lemma 2, it follows that $\Pr(X_W \in I) = 0$. Without loss of generality, take I to be the largest possible interval, which suggests that $\Pr(X_S \in [\sup I, \sup I + \epsilon)) > 0$ for any $\epsilon > 0$.

Now consider two scenarios: either the W-type player has an atom at $\sup I$ or he does not have one. If an atom exists, then the W-type player has a profitable deviation downwards, for example, from $\sup I$ to $\frac{\inf I + \sup I}{2}$. This follows from the fact that the probability of winning the prize does not change by this shift, while the realized price strictly decreases.

If, however, the W-type player does not have an atom at $\sup I$, then the S-type players have a profitable deviation from bids $x \in [\sup I, \sup I + \epsilon)$ downwards, for example, to $\frac{\inf I + \sup I}{2}$. Again, by the indifference principle, all bids produce the same expected payoff and a shift from $\sup I$ to $\frac{\inf I + \sup I}{2}$ does not entail any decrease in the winning probability, while the price strictly decreases. Thus, we conclude that this cannot be an equilibrium, and I_S is indeed a connected set. ■

8.4 Proof of Lemma 4

Proof. Assume, by contradiction, that there exists a symmetric equilibrium so that $\inf I_S > 0$, i.e., there exists $\epsilon > 0$ such that $\Pr(X_S < \epsilon) = 0$. According to Lemma 2, $\Pr(X_W < \epsilon) = 0$ as well. Moreover, recall that the S-type players have no atoms in I_S. Therefore, every S-type player who bids $\inf I_S$ receives no prize with probability 1, since both other bids (of the other S-type player and the W-type player) are above $\inf I_S$ with probability 1. This implies a strictly negative expected payoff for a bid of $\inf I_S$, and by continuity, the same holds for every neighborhood of bids close to $\inf I_S$ from above. Therefore, this cannot be an equilibrium. ■

8.5 Proof of Lemma 5

Proof. Following Corollary 1, we can assume that $\inf I_W > 0$. Since $\inf I_S = 0$, it follows that $U_S(0|F_S, F_W) = 0$ since all other bids are strictly above 0 with probability 1. Now, assume by contradiction that there exists $b \in (0, \inf I_W)$ such that $U_S(b|F_S, F_W) = c > 0$. By the fact that there are no atoms in $[0, \inf I_W)$, and by continuity of the payoff function, it follows that every S-type player can shift some probability from a neighborhood of 0 (i.e., from $[0, \epsilon)$ for some $\epsilon < b$) to b and strictly increase his payoff. Evidently, this cannot be an equilibrium, and so we conclude that $U_S(b|F_S, F_W) = 0$ for every $b \in [0, \inf I_W) \subset I_S$. By the indifference principle, the lemma follows. ■

8.6 Proof of Lemma 6

Proof. Assume, buy contradiction, that there exists a symmetric equilibrium (F_S, F_W) such that $2w_2 > w_1$ and I_W is not an interval. Note that I_S is indeed an interval, according to 3, so consider the largest interval I such that $\Pr(X_S \in I) > 0 = \Pr(I_W \in I)$ (similarly to the proof of Lemma 3). By the indifference principle, $U_S(x|F_s, F_W) = c$ for every $x \in I$. Moreover, since $\Pr(X_W \in I) = 0$, it follows that $F_W(x)$ is fixed for every $x \in I$. So, the following equality

$$U_S(x|F_s, F_W) = \left[(s_1 - 2s_2)F_S(x) + s_2\right] F_W(x) + s_2 F_S(x) - x = c,$$

implies that F_S is a linear function for every $x \in I$.

Consider $U_W(x|F_S, F_W) = (w_1 - 2w_2)F_S^2(x) + 2w_2 F_S(x) - x$. Because F_S is a linear, strictly increasing function and because $2w_2 > w_1$, if follows that $U_W(x|F_S, F_W)$ is a strictly concave function. Thus, the W-type player can strictly increase his payoff by shifting the bids from the upper and lower bounds of I to its interior. In other words, it is strictly better to choose an interior mass point in I, rather then randomizing on its boundaries, and by continuity, this also holds for close positive-probability neighborhoods of its boundaries. Thus, (F_S, F_W) cannot be an equilibrium. ■

8.7 Proof of Lemma 7

Proof. Assume, by contradiction, that there exists an equilibrium (F_S, F_W) so that $F_W(0) > 0$. Denote the expected payoff of every S-type player by C. Since the support of both S-type players are intervals starting at $x = 0$ (with no atoms), it follows that $U_W(0|F_S, F_W) = 0$ and $U_S(0|F_S, F_W) = s_2 F_W(0) = C > 0$. Thus, $C \leqslant s_2$.

Using the indifference principle, given that $U_W(x|F_S, F_W) = 0$ and $U_S(x|F_S, F_W) = C$ in every

x of the relevant supports, we deduce that

$$\begin{aligned} F_S(x) &= \frac{w_2 - \sqrt{w_2^2 - x(2w_2 - w_1)}}{2w_2 - w_1}, \quad \forall x \in [0, \overline{w}], \\ F_S(x) &= \frac{x + C - s_2}{s_1 - s_2}, \quad \forall x \in [\overline{w}, s_1 - C], \\ F_W(x) &= \frac{x - s_2 F_S(x) + C}{(s_1 - 2s_2)F_S(x) + s_2} \quad \forall x \in [0, \overline{w}]. \end{aligned}$$

Note that the function $g(x) = \frac{w_2 - \sqrt{w_2^2 - x(2w_2 - w_1)}}{2w_2 - w_1}$ is convex and $g(0) = 0$, while the function $l(x) = \frac{x+C-s_2}{s_1-s_2}$ is linear and $l(0) \leqslant 0$, with a strict inequality in case $C < s_2$. Thus, the functions $g(x)$ and $l(x)$ intersect at most twice. Denote these (potential) points by $x_1 \leqslant x_2$. Note that one of these points is $\overline{w}$ since F_S is continuous. Moreover, the fact that $g(x)$ is convex and $l(x)$ is linear implies that $l(x) > g(x)$ in every $x \in (x_1, x_2)$, while $l(x) < g(x)$ in every $x \in (0, x_1)$.

For the equilibrium to be well defined, we require that

$$F_W(x) = \frac{x - s_2 F_S(x) + C}{(s_1 - 2s_2)F_S(x) + s_2} \leqslant 1, \quad x \in [0, \overline{w}],$$

and the last inequality translates to $l(x) \leqslant g(x)$ for every $x \in [0, \overline{w}]$. Thus, we conclude that $\overline{w} = x_1$. In other words, in case the two curves intersect twice, then $\overline{w}$ is the smaller value among the two. Otherwise, in every $x \in (x_1, x_2)$, we get that $F_W(x) > 1$.

Let us now compute $\overline{w}$ through $F_S(\overline{w})$. By subtracting the following indifference conditions

$$\begin{aligned} U_S(\overline{w}|F_S, F_W) &= (s_1 - s_2)F_S(\overline{w}) + s_2 - \overline{w} = C \\ U_W(\overline{w}|F_S, F_W) &= (w_1 - 2w_2)F_S^2(\overline{w}) + 2w_2 F_S(\overline{w}) - \overline{w} = 0, \end{aligned}$$

we get $(2w_2 - w_1)F_S^2(\overline{w}) - [2w_2 - (s_1 - s_2)]F_S(\overline{w}) + s_2 - C = 0$, which yields

$$F_S(\overline{w}) = \frac{2w_2 - (s_1 - s_2) \pm \sqrt{[2w_2 - (s_1 - s_2)]^2 - 4(s_2 - C)(2w_2 - w_1)}}{2(2w_2 - w_1)}.$$

Namely, there are two possible solutions for the upper bound $\overline{w}$ derived from $F_S(\overline{w})$, and we consider only the smaller of the two.

Recall that $U_W(\overline{w}|F_S, F_W) = 0$, and for the equilibrium to hold, it must be that $U_W(x|F_S, F_W) \leqslant 0$ for every $x \in [\overline{w}, \sup I_W]$. A combination of these two conditions implies a non-positive RHS derivative $U_W'(\overline{w}|F_S, F_W) \leqslant 0$. Formally,

$$\begin{aligned} U_W(x|F_S, F_W) &= (w_1 - 2w_2)F_S^2(x) + 2w_2 F_S(x) - x \\ U_W'(\overline{w}|F_S, F_W) &= 2(w_1 - 2w_2)F_S(\overline{w})f_S(\overline{w}) + 2w_2 f_S(\overline{w}) - 1 \\ &= \frac{2(w_1 - 2w_2)}{s_1 - s_2}F_S(\overline{w}) + \frac{2w_2}{s_1 - s_2} - 1 \leqslant 0, \end{aligned}$$

where the last equality follows from the substitution $f_S(x) = \frac{1}{s_1 - s_2}$ for $x \in [\overline{w}, s_1 - C]$. Hence, we get $F_S(\overline{w}) \geqslant \frac{2w_2-(s_1-s_2)}{2(2w_2-w_1)}$. Plug-in $F_S(\overline{w})$,

$$\frac{2w_2-(s_1-s_2)-\sqrt{[2w_2-(s_1-s_2)]^2-4(s_2-C)(2w_2-w_1)}}{2(2w_2-w_1)} \geqslant \frac{2w_2-(s_1-s_2)}{2(2w_2-w_1)}$$

This inequality holds if and only if $[2w_2-(s_1-s_2)]^2-4(s_2-C)(2w_2-w_1)=0$, or equivalently, $C = s_2 - \frac{(2w_2-(s_1-s_2))^2}{4(2w_2-w_1)}$. However, the basic condition in the lemma (and in Case D) states that $s_2 - \frac{(2w_2-(s_1-s_2))^2}{4(2w_2-w_1)} < 0$, which violates the condition that $C \geqslant 0$, and the result follows. ■

8.8 Proof of Claim 1

Proof. Consider the strategy profile (F_S, F_W) given by (8). It is straightforward to verify that both CDFs are well defined. Clearly, no player can deviate to $x < 0$, nor has an incentive to bid above 4, so we consider $x \in [0, 25/9]$. For the S-type players, we get

$$\begin{aligned} U_S(x|F_S,F_W) &= (s_1-2s_2)F_S(x)F_W(x)+s_2\left[F_W(x)+F_S(x)\right]-x \\ &= -4\cdot\frac{2}{3}\cdot\frac{3x}{10}+6\left[\frac{2}{3}+\frac{3x}{10}\right]-x=4, \end{aligned}$$

while for the W-type player we get

$$\begin{aligned} U_W(x|F_S,F_W) &= (w_1-2w_2)F_S^2(x)+2w_2F_S(x)-x \\ &= 4\frac{9x^2}{100}-x \leqslant 0. \end{aligned}$$

Now, we consider $x \in [25/9, 4]$, and get

$$\begin{aligned} U_S(x|F_S,F_W) &= (s_1-2s_2)F_S(x)F_W(x)+s_2\left[F_W(x)+F_S(x)\right]-x \\ &= -4\cdot\frac{4+x-3\sqrt{x}}{6-2\sqrt{x}}\cdot\frac{\sqrt{x}}{2}+6\left[\frac{4+x-3\sqrt{x}}{6-2\sqrt{x}}+\frac{\sqrt{x}}{2}\right]-x \\ &= (6-2\sqrt{x})\frac{4+x-3\sqrt{x}}{6-2\sqrt{x}}+3\sqrt{x}-x=4. \end{aligned}$$

Thus, both S-type players are indifferent between all values of $x \in [0, 4]$. For the W-type player we get

$$\begin{aligned} U_W(x|F_S,F_W) &= (w_1-2w_2)F_S^2(x)+2w_2F_S(x)-x \\ &= 4\frac{x}{4}-x=0. \end{aligned}$$

Hence, no player has an incentive to deviate, and the given profile is indeed an equilibrium. ■

8.9 Proof of Proposition 1

Proof. Consider the strategy profile (F_S, F_W) in which $F_W(x) = 0$ and $F_S(x)$ is given by (3), and under which, the expected payoffs of all the players for a bid of $x \in [0, s_1 - s_2]$ are

$$\begin{aligned} U_S(x|F_S, F_W) &= s_1 F_S(x) + s_2[1 - F_S(x)] - x \\ &= (s_1 - s_2) \cdot \frac{x}{s_1 - s_2} + s_2 - x = s_2, \\ U_W(x|F_S, F_W) &= w_1 F_S^2(x) + 2w_2 F_S(x)[1 - F_S(x)] - x \\ &= x^2 \frac{w_1 - 2w_2}{(s_1 - s_2)^2} + x \frac{2w_2 - s_1 + s_2}{s_1 - s_2}. \end{aligned}$$

Clearly, the S-type players have no profitable deviations upwards which would induce a higher cost while the probability of winning then is identical when the bid is equal to $x = s_1 - s_2$.

Now, to see that the W-type player has no profitable deviation from $x = 0$, note that $U_W(x|F_S, F_W)$ is a quadratic function of x. For $x = s_1 - s_2$, we get $U_W(s_1 - s_2|F_S, F_W) = w_1 - s_1 + s_2 \leqslant 0$, where the inequality follows from the lemma's conditions. So, we now need to verify that the derivative of U_W at $x = 0$ is negative. Specifically, $U'_W(0|F_S, F_W) = \frac{2w_2 - s_1 + s_2}{s_1 - s_2} = \frac{2w_2}{s_1 - s_2} - 1 \leqslant 1 - 1 = 0$, and the W-type player has no profitable deviations as well, thus concluding the proof. ∎

8.10 Proof of Proposition 2

Proof. Consider the strategy profile (F_S, F_W) given by (4). We begin by showing that both functions are well-defined CDFs given that F_W is non-decreasing. Note that $F_W(0) = \frac{s_1 - w_1}{s_2} \geqslant F_S(0) = 0$, where the inequality follows from the assumption that $w_1 \geqslant s_1 - s_2$. Also, note that $F_W(w_1) = F_S(w_1) = 1$, and that one can easily verify that $F_S(x)$ is strictly increasing on $[0, w_1]$. Thus, the functions F_S and F_W are well-defined CDFs, and we can now evaluate the players' payoffs at every point x, to establish an equilibrium.

Under the given strategy profile, the expected payoff of all S-type players for a bid of $x \in [0, w_1]$ is

$$\begin{aligned} U_S(x|F_S, F_W) &= \Delta(s) F_S(x) F_W(x) + s_2 \left[F_W(x) + F_S(x)\right] - x \\ &= \Delta(s) F_S(x) \frac{x - s_2 F_S(x) + s_1 - w_1}{\Delta(s) F_S(x) + s_2} + s_2 \left[\frac{x - s_2 F_S(x) + s_1 - w_1}{\Delta(s) F_S(x) + s_2} + F_S(x)\right] - x \\ &= \frac{\Delta(s)\left[x F_S(x) - s_2 F_S^2(x)\right]}{\Delta(s) F_S(x) + s_2} + \frac{x s_2 + s_2 \Delta(s) F_S^2(x)}{\Delta(s) F_S(x) + s_2} + (s_1 - w_1) \frac{\Delta(s) F_S(x) + s_2}{\Delta(s) F_S(x) + s_2} - x \\ &= \frac{\Delta(s) x F_S(x) + x s_2}{\Delta(s) F_S(x) + s_2} + s_1 - w_1 - x = s_1 - w_1. \end{aligned}$$

Therefore, all the S-type players are indifferent between any bid $x \in [0, w_1]$, and no player has an incentive to deviate upwards above w_1. The expected payoff of the W-type player for a bid of

$x \in [0, w_1]$ is

$$\begin{aligned} U_W(x|F_S, F_W) &= [w_1 - 2w_2]\, F_S^2(x) + 2w_2 F_S(x) - x \\ &= [w_1 - 2w_2] \left[\frac{w_2 - \sqrt{w_2^2 - 2w_2 x + w_1 x}}{2w_2 - w_1} \right]^2 + 2w_2 \frac{w_2 - \sqrt{w_2^2 - 2w_2 x + w_1 x}}{2w_2 - w_1} - x \\ &= - \left[\frac{w_2^2 - 2w_2 \sqrt{w_2^2 - 2w_2 x + w_1 x} + w_2^2 - 2w_2 x + w_1 x}{2w_2 - w_1} \right] \\ &+ 2w_2 \frac{w_2 - \sqrt{w_2^2 - 2w_2 x + w_1 x}}{2w_2 - w_1} - x \\ &= \frac{2w_2 \sqrt{w_2^2 - 2w_2 x + w_1 x} + 2w_2 x - w_1 x - 2w_2 \sqrt{w_2^2 - 2w_2 x + w_1 x}}{2w_2 - w_1} - x \\ &= \frac{2w_2 x - w_1 x}{2w_2 - w_1} - x = 0. \end{aligned}$$

Thus, the W-type player has no profitable deviation, as well, and the profile is indeed an equilibrium with expected payoffs $s_1 - w_1$ and 0, as stated. ■

8.11 Proof of Proposition 3

Proof. Consider the strategy profile (F_S, F_W) given by (5). We begin by showing that both functions are well-defined CDFs, given that F_W is non-decreasing. Note that $F_W(0) = \frac{K_1}{s_2} \geqslant F_S(0) = 0$, where the inequality follows from the assumption that $K_1 \geqslant 0$. Also note that $F_S(s_1 - K_1) = 1$, and that

$$F_S(\alpha) = \frac{w_2 - \sqrt{w_2^2 + \frac{(2w_2)^2 - (s_1 - s_2)^2}{4(2w_2 - w_1)}(w_1 - 2w_2)}}{2w_2 - w_1} = \frac{w_2 - \frac{s_1 - s_2}{2}}{2w_2 - w_1} = 1 + \frac{\alpha + K_1 - s_1}{s_1 - s_2}. \tag{11}$$

Therefore, $(s_1 - s_2) F_S(\alpha) = \alpha + K_1 - s_2$. Hence,

$$F_W(\alpha) = \frac{\alpha - s_2 F_S(\alpha) + K_1}{(s_1 - s_2) F_S(\alpha) - s_2 F_S(\alpha) + s_2} = \frac{\alpha - s_2 F_S(\alpha) + K_1}{\alpha + K_1 - s_2 - s_2 F_S(\alpha) + s_2} = 1.$$

Similarly to the proof of Proposition 2, it is straightforward to verify that $F_S(x)$ is strictly increasing on $[0, w_1]$. We thus conclude that the functions F_S and F_W are well-defined CDFs, and can now evaluate the players' point-wise payoffs in order to establish an equilibrium.

Under the given strategy profile, the expected payoffs of the W-type player for a bid of $x \in [0, \alpha]$

is

$$\begin{aligned}
U_W(x|F_S,F_W) &= [w_1-2w_2]\,F_S^2(x)+2w_2F_S(x)-x \\
&= [w_1-2w_2]\left[\frac{w_2-\sqrt{w_2^2-2w_2x+w_1x}}{2w_2-w_1}\right]^2+2w_2\frac{w_2-\sqrt{w_2^2-2w_2x+w_1x}}{2w_2-w_1}-x \\
&= -\left[\frac{w_2^2-2w_2\sqrt{w_2^2-2w_2x+w_1x}+w_2^2-2w_2x+w_1x}{2w_2-w_1}\right] \\
&+ 2w_2\frac{w_2-\sqrt{w_2^2-2w_2x+w_1x}}{2w_2-w_1}-x \\
&= \frac{2w_2x-w_1x}{2w_2-w_1}-x=0.
\end{aligned}$$

Therefore, the W-type player is indifferent between any bid $x\in[0,\alpha]$. In addition, a bid of $x\in(\alpha,s_1-K_1]$ would produce a negative payoff for the W-type player as $[F_S(x)-1](s_1-s_2)+s_1-K_1=x$ and

$$\begin{aligned}
U_W(x|F_S,F_W) &= [w_1-2w_2]\,F_S^2(x)+2w_2F_S(x)-[F_S(x)-1](s_1-s_2)-s_1+K_1 \\
&= \Delta(w)t^2+(2w_2-\Delta(s'))t-s_2+K_1,
\end{aligned}$$

where $t=F_S^2(x)$, $\Delta(w)=w_1-2w_2$, and $\Delta(s')=s_1-s_2$. Denote $H(t)=\Delta(w)t^2+(2w_2-\Delta(s'))t-s_2+K_1$, which is a parabolic function with a unique maximum point (by the assumption that $\Delta(w)<0$) and $H(F_S(\alpha))=0$. Moreover, $H'(t)=2\Delta(w)t+(2w_2-\Delta(s'))$ and

$$H'(F_S(\alpha))=2\Delta(w)F_S(\alpha)+(2w_2-\Delta(s'))=-2\left[w_2-\tfrac{s_1-s_2}{2}\right]+(2w_2-\Delta(s'))=0,$$

where the second equality follows from Equation (11). Thus, $H(t)$ is decreasing for every $t>F_S(\alpha)$, which implies that $U_W(x|F_S,F_W)<0$ for every $x>\alpha$, as needed. Thus, the W-type player has no incentive to deviate upwards, above α.

We now consider the S-type players. Denote $\Delta(s)=s_1-2s_2$. The expected payoff of the S-type players for a bid of $x\in[0,\alpha]$ is

$$\begin{aligned}
U_S(x|F_S,F_W) &= \Delta(s)F_S(x)F_W(x)+s_2\left[F_W(x)+F_S(x)\right]-x \\
&= \Delta(s)F_S(x)\frac{x-s_2F_S(x)+K_1}{\Delta(s)F_S(x)+s_2}+s_2\left[\frac{x-s_2F_S(x)+K_1}{\Delta(s)F_S(x)+s_2}+F_S(x)\right]-x \\
&= \frac{\Delta(s)\left[xF_S(x)-s_2F_S^2(x)\right]}{\Delta(s)F_S(x)+s_2}+\frac{xs_2+s_2\Delta(s)F_S^2(x)}{\Delta(s)F_S(x)+s_2}+K_1\frac{\Delta(s)F_S(x)+s_2}{\Delta(s)F_S(x)+s_2}-x \\
&= \frac{\Delta(s)xF_S(x)+xs_2}{\Delta(s)F_S(x)+s_2}+K_1-x=K_1,
\end{aligned}$$

and for a bid of $x \in [\alpha, s_1 - K_1]$, the expected payoff is

$$\begin{aligned} U_S(x|F_S, F_W) &= \Delta(s)F_S(x) + s_2\left[1 + F_S(x)\right] - x \\ &= \Delta(s')F_S(x) + s_2 - x \\ &= \Delta(s')\left[1 + \frac{x + K_1 - s_1}{\Delta(s')}\right] + s_2 - x \\ &= \Delta(s') + x + K_1 - s_1 + s_2 - x = K_1. \end{aligned}$$

Thus, no player has a profitable deviation, and the stated profile is indeed an equilibrium, with expected payoffs of K_1 and 0, as needed. ■

8.12 Proof of Proposition 4

Proof. Consider the strategy profile (F_S, F_W) given by (6). We begin by showing that both functions are well-defined CDFs, given that F_W is non-decreasing. Note that $F_S(0) = 0 < F_S(s_1) = 1$ and F_S is strictly increasing in $[0, s_1]$. Also note that

$$\begin{aligned} \alpha_1 &= s_2 \frac{2w_2 - s_2 - \sqrt{(2w_2 - s_2)^2 + 4K_1(2w_2 - w_1)}}{2(2w_2 - w_1)} \\ &= s_2 \frac{2w_2 - s_2 - \sqrt{(2w_2 - s_2)^2 - 4\left(\frac{[2w_2 - (s_1 - s_2)]^2}{4(2w_2 - w_1)} - s_2\right)(2w_2 - w_1)}}{2(2w_2 - w_1)} \\ &= s_2 \frac{2w_2 - s_2 - \sqrt{2s_1 s_2 - s_1^2 + 4s_1 w_2 - 4s_2 w_1}}{2(2w_2 - w_1)} \\ &\leqslant s_2 \frac{2w_2 - s_2 - \sqrt{2s_1 s_2 - s_1^2 + 2s_1(s_1 - s_2) - 4s_2(s_1 - s_2)}}{2(2w_2 - w_1)} \\ &= s_2 \frac{2w_2 - s_2 - \sqrt{s_1^2 - 4s_2 s_1 + 4s_2^2}}{2(2w_2 - w_1)} \\ &= s_2 \frac{2w_2 - s_2 - (s_1 - 2s_2)}{2(2w_2 - w_1)} = s_2 \frac{2w_2 - (s_1 - s_2)}{2(2w_2 - w_1)} \leqslant s_2 \cdot 1 < \alpha_2, \end{aligned}$$

as needed. Moreover, we can show that the proposition's conditions imply that $\alpha_1 \geqslant 0$ (i.e., $2w_2 \geqslant s_2$), and it is a straightforward to verify that F_S is continuous, specifically at $x = \alpha_1, \alpha_2$. Therefore, we can conclude that both functions are well defined.

Let us now verify that the profile of strategies which consists of F_W and F_S constitutes an equilibrium. We begin with the W-type player. For $x \in [\alpha_1, \alpha_2]$ we get

$$\begin{aligned}
U_W(x|F_S, F_W) &= [w_1 - 2w_2]\, F_S^2(x) + 2w_2 F_S(x) - x \\
&= [w_1 - 2w_2] \left[\frac{w_2 - \sqrt{w_2^2 - (x + K_2)(2w_2 - w_1)}}{2w_2 - w_1} \right]^2 \\
&+ 2w_2 \frac{w_2 - \sqrt{w_2^2 - (x - K_1)(2w_2 - w_1)}}{2w_2 - w_1} - x \\
&= -\frac{2w_2^2 - (x + K_2)(2w_2 - w_1) - 2w_2\sqrt{w_2^2 - (x + K_2)(2w_2 - w_1)}}{2w_2 - w_1} \\
&+ 2w_2 \frac{w_2 - \sqrt{w_2^2 - (x + K_2)(2w_2 - w_1)}}{2w_2 - w_1} - x \\
&= \frac{(x + K_2)(2w_2 - w_1)}{2w_2 - w_1} - x = -K_1.
\end{aligned}$$

Therefore, the W-type player is indifferent between all values of $x \in [\alpha_1, \alpha_2]$ that produce an expected payoff of $-K_1 < 0$.

Now consider $x \in [0, \alpha_1)$,

$$\begin{aligned}
U_W(x|F_S, F_W) &= [w_1 - 2w_2]\, F_S^2(x) + 2w_2 F_S(x) - x \\
&= [w_1 - 2w_2]\, \frac{x^2}{s_2^2} + 2w_2 \frac{x}{s_2} - x.
\end{aligned}$$

Thus, for $x \in [0, \alpha_1)$, the function $U_W(x|F_S, F_W)$ is parabolic with $U_W(0|F_S, F_W) = 0$, $U_W'(0|F_S, F_W) \geqslant 0$ (which follows from $2w_2 > s_1 - s_2 \geqslant w_1$ and $-K_1 > 0$) and

$$\begin{aligned}
U_W'(\alpha_1|F_S, F_W) &= 2\,[w_1 - 2w_2]\, \frac{\alpha_1}{s_2^2} + \frac{2w_2}{s_2} - 1 \\
&= 2\,[w_1 - 2w_2]\, s_2 \frac{2w_2 - s_2 - \sqrt{(2w_2 - s_2)^2 - 4K_2(2w_2 - w_1)}}{2(2w_2 - w_1)s_2^2} + \frac{2w_2}{s_2} - 1 \\
&= -\frac{2w_2 - s_2 - \sqrt{(2w_2 - s_2)^2 + 4K_1(2w_2 - w_1)}}{s_2} + \frac{2w_2 - s_2}{s_2} \\
&= \frac{\sqrt{(2w_2 - s_2)^2 + 4K_1(2w_2 - w_1)}}{s_2} \geqslant 0.
\end{aligned}$$

Since $U_W(\alpha_1|F_S, F_W) = -K_1$ and $U_W(x|F_S, F_W)$ is increasing for $x \in [0, \alpha_1)$, we conclude that $U_W(x|F_S, F_W) \leqslant K_2$ for every $x \in [0, \alpha_1)$, and that there exists no profitable deviation downwards for the W-type player.

We now consider $x \in (\alpha_2, s_1]$.

$$\begin{aligned}
U_W(x|F_S, F_W) &= [w_1 - 2w_2]\, F_S^2(x) + 2w_2 F_S(x) - x \\
&= [w_1 - 2w_2]\, \frac{(x - s_2)^2}{(s_1 - s_2)^2} + 2w_2 \frac{x - s_2}{s_1 - s_2} - x.
\end{aligned}$$

So,

$$\begin{aligned} U'_W(\alpha_2|F_S,F_W) &= 2\left[w_1-2w_2\right]\frac{\left(s_2+(s_1-s_2)\frac{2w_2-(s_1-s_2)}{2(2w_2-w_1)}-s_2\right)}{(s_1-s_2)^2}+\frac{2w_2}{s_1-s_2}-1 \\ &= -\frac{2w_2-(s_1-s_2)}{s_1-s_2}+\frac{2w_2}{s_1-s_2}-1=0, \end{aligned}$$

while $U_W(\alpha_2|F_S,F_W)=-K_1$, and $U_W(s_1|F_S,F_W)=w_1-s_1<0$. Therefore, we can conclude that $U_W(x|F_S,F_W)\leqslant -K_1$ for every $x\in(\alpha_2,s_1]$, as needed. Therefore, we have established that the W-type player has no profitable deviations.

We now consider the S-type players. For $x\in[0,\alpha_1)$, we get

$$\begin{aligned} U_S(x|F_S,F_W) &= (s_1-2s_2)F_S(x)F_W(x)+s_2\left[F_W(x)+F_S(x)\right]-x \\ &= (s_1-2s_2)F_S(x)\cdot 0+s_2\left[0+\frac{x}{s_2}\right]-x=0, \end{aligned}$$

whereas, for $x\in(\alpha_2,s_1]$, we get

$$\begin{aligned} U_S(x|F_S,F_W) &= (s_1-2s_2)F_S(x)F_W(x)+s_2\left[F_W(x)+F_S(x)\right]-x \\ &= (s_1-2s_2)\frac{x-s_2}{s_1-s_2}\cdot 1+s_2\left[1+\frac{x-s_2}{s_1-s_2}\right]-x \\ &= (s_1-2s_2)\frac{x-s_2}{s_1-s_2}+s_2\frac{x+s_1-2s_2}{s_1-s_2}-x=0, \end{aligned}$$

Therefore, in these intervals, the S-type players get an expected payoff of 0 for every bid. In addition, for $x\in[\alpha_1,\alpha_2]$,

$$\begin{aligned} U_S(x|F_S,F_W) &= (s_1-2s_2)F_S(x)\frac{x-s_2F_S(x)}{(s_1-2s_2)F_S(x)+s_2}+s_2\left[\frac{x-s_2F_S(x)}{(s_1-2s_2)F_S(x)+s_2}+F_S(x)\right]-x \\ &= \left[(s_1-2s_2)F_S(x)+s_2\right]\frac{x-s_2F_S(x)}{(s_1-2s_2)F_S(x)+s_2}+s_2F_S(x)-x=0. \end{aligned}$$

Hence, we can conclude that the S-type players have an expected payoff of 0 for every $x\in[0,s_1]$, and that there are no profitable deviations, thus establishing an equilibrium. ■

8.13 Proof of Proposition 5

Proof. Consider the strategy profile (F_S,F_W) given by (7). Note that both functions are well-defined CDFs, given that F_S is non-decreasing. Specifically, $F_W(0)=F_S(\alpha)=0<F_W(w_1)=F_S(w_1)=1$, and F_W is strictly increasing and continuous (by the choice of α) in $[0,w_1]$.

We now verify that the profile of strategies (F_W,F_S) constitutes an equilibrium. We begin with the W-type players. For $x\in[0,\alpha]$, we get

$$\begin{aligned} U_W(x|F_S,F_W) &= \left[(w_1-2w_2)F_W(x)+w_2\right]F_S(x)+w_2F_W(x)-x \\ &= \left[(w_1-2w_2)\tfrac{x}{w_2}+w_2\right]\cdot 0+w_2\tfrac{x}{w_2}-x=0, \end{aligned}$$

and for $x \in [\alpha, w_1]$, we get

$$\begin{aligned} U_W(x|F_S, F_W) &= [(w_1 - 2w_2)F_W(x) + w_2]\, F_S(x) + w_2 F_W(x) - x \\ &= [(w_1 - 2w_2)F_W(x) + w_2]\, \frac{x - w_2 F_W(x)}{(w_1 - 2w_2)F_W(x) + w_2} + w_2 F_W(x) - x = 0. \end{aligned}$$

Hence, the W-type players are indifferent between all values of $x \in [0, w_1]$ which produce an expected payoff of 0.

We now consider the S-type player. For $x \in [\alpha, w_1]$, we get

$$\begin{aligned} U_S(x|F_S, F_W) &= \Delta(s)F_W^2(x) + 2s_2 F_W(x) - x \\ &= \Delta(s)\left[\frac{-s_2 + \sqrt{s_2^2 + \Delta(s)(s_1 - w_1 + x)}}{\Delta(s)}\right]^2 \\ &+ 2s_2 \frac{-s_2 + \sqrt{s_2^2 + \Delta(s)(s_1 - w_1 + x)}}{\Delta(s)} - x \\ &= \frac{s_2^2 - 2s_2\sqrt{s_2^2 + \Delta(s)(s_1 - w_1 + x)} + s_2^2 + \Delta(s)(s_1 - w_1 + x)}{\Delta(s)} \\ &+ 2s_2 \frac{-s_2 + \sqrt{s_2^2 + \Delta(s)(s_1 - w_1 + x)}}{\Delta(s)} - x \\ &= \frac{\Delta(s)(s_1 - w_1 + x)}{\Delta(s)} - x = s_1 - w_1, \end{aligned}$$

therefore, the expected payoff of the S-type player is $s_1 - w_1$ for every $x \in [\alpha, w_1]$. In addition, we consider $x \in [0, \alpha]$, and note that $U_S(x|F_S, F_W)$ constitutes the following parabolic function,

$$\begin{aligned} U_S(x|F_S, F_W) &= (s_1 - 2s_2)F_W^2(x) + 2s_2 F_W(x) - x \\ &= (s_1 - 2s_2)\frac{x^2}{w_2^2} + 2s_2 \frac{x}{w_2} - x. \end{aligned}$$

By differentiating and inserting in $x = \alpha$, we get

$$\begin{aligned} U_S'(\alpha|F_S, F_W) &= \Delta(s)\frac{2\alpha}{w_2^2} + \frac{2s_2}{w_2} - 1 \\ &= \Delta(s)\frac{2\frac{w_2}{2\Delta(s)}\left[-2s_2 + w_2 + \sqrt{(2s_2 - w_2)^2 + 4\Delta(s)(s_1 - w_1)}\right]}{w_2^2} + \frac{2s_2}{w_2} - 1 \\ &= \frac{-2s_2 + w_2 + \sqrt{(2s_2 - w_2)^2 + 4\Delta(s)(s_1 - w_1)}}{w_2} + \frac{2s_2}{w_2} - 1 > 0, \end{aligned}$$

As such, the function is increasing for x below and sufficiently close to α. Combining this result with the fact that $U_S(0|F_S, F_W) = 0$, we conclude that $U_S(x|F_S, F_W) < U_S(\alpha|F_S, F_W) = s_1 - w_1$ for $x \in [0, \alpha]$, and that the S-type player does not have a profitable deviation downwards. To conclude, we have shown that there are no profitable deviations for any of the players, thus establishing an equilibrium. ■

8.14 Proof of Proposition 6

Proof. Consider the strategy profile (F_S, F_W) where $F_W(x) = 0$ and $F_S(x)$ is given by (9). Fix $F_W = \mathbf{1}_{\{x \geqslant 0\}}$ so that the W-type player always bids $x = 0$. Given some CDF F_S with no atoms in $[0, s_1)$, the W-type player has an expected payoff of 0, whereas an S-type player who bids x has an expected payoff of

$$U_S(x|F_S, F_W) = s_1 F_S^{n-2}(x) + s_2(n-2)F_S^{n-3}(x)(1 - F_S(x)) - x.$$

Now, we fix F_S such that $U_S(x|F_S, F_W) = 0$ for every $x \in [0, s_1]$. Note that this CDF is well defined since $F_S(x) = 0$ for every $x \leqslant 0$, $F_S(x) = 1$ for every $x \geqslant s_1$, and the function is strictly increasing in the given interval.

To show that (F_S, F_W) is an equilibrium, we consider a unilateral deviation of some player, either of type W or type S. An S-type player has no profitable deviation for a bid $x \in [0, s_1]$ since all bids generate a payoff of zero. In addition, any deviation upwards to $x > s_1$ entails a negative expected payoff. Thus, we can focus on a deviation of an W-type player.

Assume that the W-type player bids $x > 0$, and that $[s_1 - (n-2)s_2] \geqslant \max\{w_1, (n-1)w_2\}$. According the Eq. (2), the player's expected payoff would be

$$\begin{aligned}
U_W(x|F_S, F_W) &= w_1 F_S^{n-1}(x) + w_2(n-1)(1 - F_S(x))F_S^{n-2}(x) - x \\
&\leqslant [s_1 - (n-2)s_2]F_S^{n-1}(x) + [s_1 - (n-2)s_2][1 - F_S(x)]F_S^{n-2}(x) - x \\
&= (s_1 - (n-2)s_2)F_S^{n-2}(x) - x \\
&< s_1 F_S^{n-2}(x) + s_2(n-2)F_S^{n-3}(x)(1 - F_S(x)) - x \\
&= U_S(x|F_S, F_W) = 0,
\end{aligned}$$

where the first inequality follows from the condition $[s_1 - (n-2)s_2] \geqslant \max\{w_1, (n-1)w_2\}$, and the second inequality follows from the fact that $s_2(n-2)F^{n-3}(x) > 0$ for $x > 0$. Otherwise, assume that $(n-2)s_2 \geqslant (n-1)w_2$ and recall that $s_1 > w_1$. Then,

$$\begin{aligned}
U_W(x|F_S, F_W) &= w_1 F_S^{n-1}(x) + w_2(n-1)(1 - F_S(x))F_S^{n-2}(x) - x \\
&< s_1 F_S^{n-2}(x) + s_2(n-2)(1 - F_S(x))F_S^{n-3}(x) - x \\
&= U_S(x|F_S, F_W) = 0,
\end{aligned}$$

where the inequality follows from our preliminary assumptions, $(n-2)s_2 \geqslant (n-1)w_2$ and $s_1 > w_1$, along with the fact that $F_S(x) \leqslant 1$. We conclude that the W-type player has no profitable deviation upwards, and (F_S, F_W) is indeed an equilibrium. ■

8.15 Proof of Proposition 7

Proof. Consider the strategy profile (F_S, F_W) given by (10). We begin by showing that the functions F_W and F_S are well-defined CDFs, given that F_S is non decreasing in $[\alpha_1, w_1]$. For that purpose, we first need to prove that α_1 and $G(x)$ are well-defined. Consider the equation

$$s_1 - w_1 + \alpha_1 = s_1 \left[\tfrac{\alpha_1}{w_2}\right]^{(n-1)/(n-2)} + s_2(n-2)\tfrac{\alpha_1}{w_2}\left[1 - \left[\tfrac{\alpha_1}{w_2}\right]^{1/(n-2)}\right].$$

If we substitute α_1 with 0, then the LHS is strictly greater than the RHS. However, for $\alpha_1 = w_2$, we obtain the reverse inequality. Thus, by the Mean-Value Theorem (MVT), there exists a solution $\alpha_1 \in [0, w_1]$. Similarly, for every $x \in (\alpha_1, w_1)$, we can take the equation

$$s_1 - w_1 + x = s_1 G^{n-1}(x) + s_2(n-2)G^{n-2}(x)[1 - G(x)],$$

and substitute $G(x)$ with 0 and 1. Again, we get reverse inequalities (between the two cases), and the MVT ensures that a solution $G(x)$ exists. Note that for $x = w_1$ we get $G(w_1) = 1$, and for $x = \alpha_1$ both equations coincide so that $G(\alpha_1) = \left[\tfrac{\alpha_1}{w_2}\right]^{\frac{1}{n-2}}$. Thus, α_1 and $G(x)$ are well-defined, and F_W is continuous, thus implying that F_S is continuous, as well. By differentiating both sides of the second equation, we get

$$G'(x) = \frac{1}{G^{n-3}(x)\left[G(x)[s_1(n-1) - s_2(n-2)(n-1)] + s_2(n-2)^2\right]} \geqslant 0, \quad \forall G(x) \in (0,1].$$

Therefore, $G(x)$ is non-decreasing. We conclude that both functions, F_W and F_S, are well-defined CDFs, as needed.

We next establish an equilibrium, beginning with the single S-type player. Taking the expected payoff of the single S-type player and inserting in F_W for $x \in [\alpha_1, w_1]$, we get

$$U_S(x|F_S, F_W) = s_1 G^{n-1}(x) + s_2(n-2)G^{n-2}(x)[1 - G(x)] - x = s_1 - w_1,$$

where the equality follows from the definition of $G(x)$. To evaluate a possible deviation of the S-type player downwards to $x \in [0, \alpha_1)$, consider the functions

$$\begin{aligned} U_S(x|F_S, F_W) &= [s_1 - s_2(n-2)]F_W^{n-1}(x) + s_2(n-2)F_W^{n-2}(x) - x \\ &= [s_1 - s_2(n-2)] \cdot \left[\frac{x}{w_2}\right]^{\frac{n-1}{n-2}} + \frac{s_2(n-2)x}{w_2} - x \\ \frac{dU_S(x|F_S, F_W)}{dx} &= [s_1 - s_2(n-2)] \cdot \frac{n-1}{(n-2)x} \cdot \left[\frac{x}{w_2}\right]^{\frac{n-1}{n-2}} + \frac{s_2(n-2)}{w_2} - 1. \end{aligned}$$

Since $s_1 \geqslant s_2(n-2)$, it follows that U'_S is non-decreasing for $x \in [0, \alpha_1)$. In other words, the monotonicity of U'_S implies that U_S is convex with no interior maxima in $x \in [0, \alpha_1)$. Since $U_S(0|F_S, F_W) = 0 < s_1 - w_1 = U_S(\alpha_1|F_S, F_W)$, we conclude that $U_S(x|F_S, F_W) < U_S(\alpha_1|F_S, F_W)$ for every $x \in [0, \alpha_1)$, which implies that the S-type player has no profitable deviations downwards.

For the W-type players, the expected payoff is given by

$$U_W(x|F_S, F_W) = w_1 F_W^{n-2}(x)F_S(x) + w_2\left[(1 - F_S(x))F_W^{n-2}(x) + (n-3)F_W^{n-3}(x)F_S(x)(1 - F_W(x))\right] - x.$$

For $x \in [0, \alpha_1]$ we get

$$\begin{aligned} U_W(x|F_S, F_W) &= w_1 F_W^{n-2}(x) \cdot 0 + w_2 \left[(1-0)F_W^{n-2}(x) + (n-3)F_W^{n-3}(x) \cdot 0 \cdot (1 - F_W(x))\right] - x \\ &= w_2 F_W^{n-2}(x) - x \\ &= w_2 \frac{x}{w_2} - x = 0. \end{aligned}$$

For $x \in [\alpha_1, w_1]$, we can see that F_S is specifically defined under the condition that $U_W = (x|F_S, F_W) = 0$. Therefore, again, no player has a profitable deviation, and (F_S, F_W) is an equilibrium as stated. ■

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